

CONSTRUCTION OF AN INVARIANT FOR INTEGRAL HOMOLOGY 3-SPHERES VIA COMPLETED KAUFFMAN BRACKET SKEIN ALGEBRAS

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ABSTRACT. We construct an invariant $z(M) = 1 + a_1(A^4 - 1) + a_2(A^4 - 1)^2 + a_3(A^4 - 1)^3 + \dots \in \mathbb{Q}[[A^4 - 1]] = \mathbb{Q}[[A + 1]]$ for an integral homology 3-sphere M using a completed skein algebra and a Heegaard splitting. The invariant $z(M) \bmod ((A + 1)^{n+1})$ is a finite type invariant of order n . In particular, $-a_1/6$ equals the Casson invariant. If M is the Poincaré homology 3-sphere, a natural homomorphism maps the unified WRT invariant of M defined by Habiro [5] to $z(M) \bmod ((A + 1)^{14})$.

1. INTRODUCTION

Heegaard splitting theory clarifies a relationship between mapping class groups on surfaces and closed oriented 3-manifolds. In particular, there exists some equivalence relation \sim of Torelli groups of a surface $\Sigma_{g,1}$ with genus g and non-empty connected boundary, and the well-defined bijective map

$$\lim_{g \rightarrow \infty} \mathcal{I}(\Sigma_{g,1}) / \sim \rightarrow \mathcal{H}(3)$$

plays an important role, where we denote by $\mathcal{I}(\Sigma_{g,1})$ the Torelli group of $\Sigma_{g,1}$ and by $\mathcal{H}(3)$ the set of integral homology 3-spheres, i.e. closed oriented 3-manifolds whose homology groups are isomorphic to the homology group of S^3 . For details, see Fact 2.2 in this paper. This bijective map makes it possible to study integral homology 3-spheres using the structure of Torelli groups. See, for example, Morita [10] and Pitsch [12] [13].

On the other hand, in our previous papers [14] [15] [16], we study some new relationship between the Kauffman bracket skein algebra and the mapping class group of a surface. It gives us a new way of studying the mapping class group. For example [16], we reconstruct the first Johnson homomorphism in terms of the skein algebra. Since the Kauffman bracket skein algebra comes from link theory, we expect that this relationship brings us a new information of 3-manifolds.

The aim of this paper is to construct an invariant $z(M)$ for an integral homology 3-sphere M using completed skein algebras and the above bijective map. In other words, the aim of this paper is to prove the following main theorem.

Theorem 1.1 (Theorem 3.3). *The map $Z : \mathcal{I}(\Sigma_{g,1}) \rightarrow \mathbb{Q}[[A + 1]]$ defined by*

$$Z(\xi) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{(-A + A^{-1})^i i!} e_*((\zeta(\xi))^i)$$

induces an invariant

$$z : \mathcal{H}(3) \rightarrow \mathbb{Q}[[A + 1]], M(\xi) \rightarrow Z(\xi),$$

where e_* is the $\mathbb{Q}[[A+1]]$ -module homomorphism induced by standard embedding. Here $\zeta : \mathcal{I}(\Sigma_{g,1}) \rightarrow \widehat{\mathcal{S}(\Sigma_{g,1})}$ is an embedding defined in Theorem 3.2.

We remark we do not rely on number theory for constructing the invariant.

Let V be a \mathbb{Q} -vector space. In our paper, a map $z' : \mathcal{H}(3) \rightarrow V$ is called a finite type invariant of order n if and only if the \mathbb{Q} -linear map $z' : \mathbb{Q}\mathcal{H}(3) \rightarrow V$ induced by $z' : \mathcal{H}(3) \rightarrow V$ satisfies the condition that

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} z'(M(\prod_{i=1}^{2n+2} \xi_i^{\epsilon_i})) = 0.$$

for any $\xi_1, \xi_2, \dots, \xi_{2n+2} \in \mathcal{I}(\Sigma_{g,1})$. The above condition and the condition that

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} z'(M(\prod_{i=1}^{n+1} \xi_i^{\epsilon_i})) = 0.$$

for any $\xi_1, \xi_2, \dots, \xi_{n+1} \in \mathcal{K}(\Sigma_{g,1})$ are equivalent to each other. This follows from [2] Theorem 1 and [3] subsection 1.8. Furthermore, in our paper, a finite type invariant $z' : \mathcal{H}(3) \rightarrow V$ of order n is called nontrivial if and only if the \mathbb{Q} -linear map $z' : \mathbb{Q}\mathcal{H}(3) \rightarrow V$ induced by $z' : \mathcal{H}(3) \rightarrow V$ satisfies the condition that there exists $\xi_1, \xi_2, \dots, \xi_{2n} \in \mathcal{I}(\Sigma_{g,1})$ such that

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} z'(M(\prod_{i=1}^{2n} \xi_i^{\epsilon_i})) \neq 0.$$

By [2] Theorem 1 and [3] subsection 1.8, the above condition and the condition that there exists $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{K}(\Sigma_{g,1})$ such that

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} z'(M(\prod_{i=1}^n \xi_i^{\epsilon_i})) \neq 0.$$

are equivalent to each other. The invariant $z : \mathcal{H}(3) \rightarrow \mathbb{Q}[[A+1]]$ defined in this paper induces a finite type invariant $z(M) \in \mathbb{Q}[[A+1]]/((A+1)^n)$ of order $n+1$ for $M \in \mathcal{H}(3)$ (Corollary 3.12). In Proposition 3.13, we prove the finite type invariant $z(M) \in \mathbb{Q}[[A+1]]/((A+1)^n)$ of order $n+1$ is nontrivial, where we use a connected sum of the Poincaré spheres.

Furthermore, we give some computations of this invariant z for some integral homology 3-spheres. As a corollary of this computation, the coefficient of $(A^4 - 1)$ in z is (-6) times the Casson invariant. On the other hand, Habiro [5] defined the unified WRT invariant $\tau : \mathcal{H}(3) \rightarrow \widehat{\mathbb{Z}[q]}$. Here $\widehat{\mathbb{Z}[q]}$ is

$$\widehat{\mathbb{Z}[q]} \stackrel{\text{def.}}{=} \varprojlim_{i \rightarrow \infty} \mathbb{Z}[q]/((q)_i)$$

where $(q)_i = (1-q)(1-q^2) \cdots (1-q^i)$. We remark there exists a natural map $\alpha : \widehat{\mathbb{Z}[q]} \rightarrow \mathbb{Z}[[q-1]] \hookrightarrow \mathbb{Q}[[q-1]]$. If M is the Poincaré homology 3-sphere, $z(M) \bmod ((A+1)^{14})$ is equal to $\alpha \circ \tau(M)|_{q=A^4} \bmod ((A+1)^{14})$. This lead us the following.

Expectation 1.2. *The invariant z is equal to the unified WRT invariant $\alpha \circ \tau$, in other words, we have $z(M) = \alpha \circ \tau(M)|_{q=A^4}$ for any $M \in \mathcal{H}(3)$.*

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2. MAPPING CLASS GROUPS AND CLOSED 3-MANIFOLDS

Let Σ_g denote an closed oriented surface of genus g standardly embedded in the oriented 3-sphere S^3 . The embedded surface Σ_g separates S^3 into two handle bodies of genus g , $S^3 = H_g^+ \cup_\varphi H_g^-$ where $\varphi : \Sigma_g = \partial H_g^+ \rightarrow \partial H_g^-$ is a diffeomorphism. We fix an open disk D in Σ_g and denote $\Sigma_{g,1} \stackrel{\text{def.}}{=} \Sigma_g \setminus D$. The embedding $\Sigma_g \hookrightarrow S^3$ determines two natural subgroups of

$$\mathcal{M}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} \text{Diff}^+(\Sigma_{g,1}, \partial\Sigma_{g,1}) / \text{Diff}_0(\Sigma_{g,1}, \partial\Sigma_{g,1}),$$

namely

$$\mathcal{M}(H_{g,1}^\epsilon) \stackrel{\text{def.}}{=} \text{Diff}^+(H_g^\epsilon, D) / \text{Diff}_0(H_g^\epsilon, D).$$

for $\epsilon \in \{+, -\}$. We denote $M(\xi) \stackrel{\text{def.}}{=} H_g^+ \cup_{\varphi \circ \xi} H_g^-$. Let $\mathcal{I}(\Sigma_{g,1})$ be the Torelli group of the surface $\Sigma_{g,1}$, which is the set consisting of all elements of $\mathcal{M}(\Sigma_{g,1})$ acting trivially on $H_1(\Sigma_{g,1})$. We remark that there is a natural injective stabilization map $\mathcal{M}(\Sigma_{g,1}) \hookrightarrow \mathcal{M}(\Sigma_{g+1,1})$, which is compatible with the definitions of the above two subgroups.

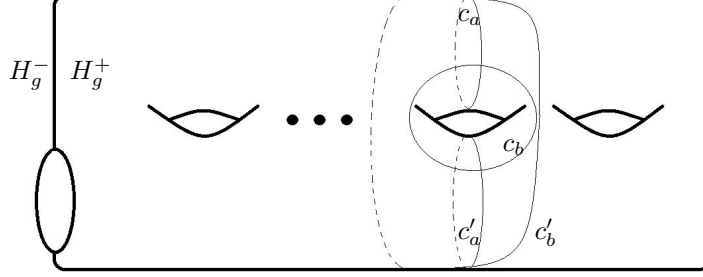
Definition 2.1. For ξ_1 and $\xi_2 \in \mathcal{I}(\Sigma_{g,1})$, we define $\xi_1 \sim \xi_2$ if there exist $\eta^+ \in \mathcal{M}(H_{g,1}^+)$ and $\eta^- \in \mathcal{M}(H_{g,1}^-)$ satisfying $\xi_1 = \eta^- \xi_2 \eta^+$.

Fact 2.2 (For example, see [10] [12][13]). The map

$$\varinjlim_{g \rightarrow \infty} (\mathcal{I}(\Sigma_{g,1}) / \sim) \rightarrow \mathcal{H}(3), \xi \mapsto M(\xi)$$

is bijective, where $\mathcal{H}(3)$ is the set of integral homology 3-spheres, i.e., closed oriented 3-manifolds whose homology group is isomorphic to the homology group of S^3 .

We denote $\mathcal{IM}(H_{g,1}^\epsilon) \stackrel{\text{def.}}{=} \mathcal{I}(\Sigma_{g,1}) \cap \mathcal{M}(H_{g,1}^\epsilon)$ for $\epsilon \in \{+, -\}$.

FIG 1. c_a, c'_a, c_b and c'_b

Lemma 2.3 (Pitsch [13], Theorem 9, P.295, Omori [11]). *For $\epsilon \in \{+, -\}$, the subgroup $\mathcal{IM}(H_{g,1}^\epsilon)$ is generated by*

$$\{t_{\xi(c_a)\xi(c'_a)}|\xi \in \mathcal{M}(H_{g,1}^\epsilon)\} \cup \{t_{\xi(c_b)\xi(c'_b)}|\xi \in \mathcal{M}(H_{g,1}^\epsilon)\},$$

where the simple closed curves c_a, c'_a, c_b and c'_b are as in Figure 1.

Proof. We prove the lemma in the case ϵ is $+$. Let $IAut\pi_1(H_g^+, *)$ be the kernel of $Aut\pi_1(H_g^+, *) \rightarrow Aut(H_1(H_g^+))$. By [8], Theorem N4, p.168, $IAut\pi_1(H_g^+, *)$ is generated by

$$\{x_* \in IAut\pi_1(H_g^+, *) | x \in \{t_{\xi(c_b)\xi(c'_b)}|\xi \in \mathcal{M}(H_{g,1}^+)\}\},$$

where we denote by x_* the element of $IAut\pi_1(H_g^+, *)$ induced by $x \in \{t_{\xi(c_b)\xi(c'_b)}|\xi \in \mathcal{M}(H_{g,1}^+)\}$. We denote by $\mathcal{LIM}(H_{g,1}^+)$ the Luft-Torelli group which is the kernel of $\mathcal{IM}(H_{g,1}^+) \rightarrow IAut\pi_1(H_g^+, *)$. Pitsch [13], Theorem 9, P.295 proves that $\mathcal{LIM}(H_{g,1}^+)$ is generated by

$$\{t_{\xi(c_a)\xi(c'_a)}|\xi \in \mathcal{M}(H_{g,1}^+)\}.$$

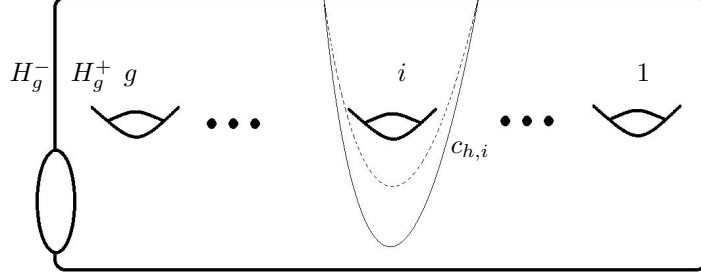
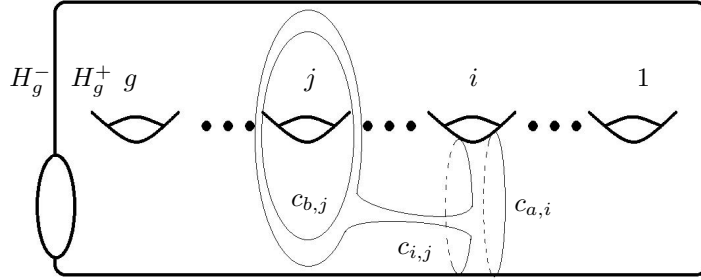
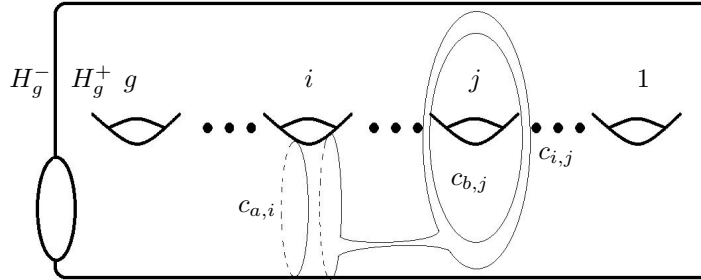
This proves the case that ϵ is $+$. If ϵ is $-$, replacing a by b , the same proof works. This finishes the proof. \square

Lemma 2.4 ([13], Lemma 4, p.285). *Let G be a subgroup of $\mathcal{M}(H_{g,1}^+) \cap \mathcal{M}(H_{g,1}^-)$ such that the natural map $G \rightarrow Aut(H_1(H_{g,1}^+))$ is onto. For two elements ξ_1 and $\xi_2 \in \mathcal{I}(\Sigma_{g,1})$, $\xi_1 \sim \xi_2$ if and only if there exist $\eta_G \in G$, $\eta^+ \in \mathcal{IM}(H_{g,1}^+)$ and $\eta^- \in \mathcal{IM}(H_{g,1}^-)$ satisfying $\eta^- \eta_G \xi_1 \eta_G^{-1} \eta^+ = \xi_2$.*

Proof. Pitsch proved the above claim in the case $G = \mathcal{M}(H_{g,1}^+) \cap \mathcal{M}(H_{g,1}^-)$. The proof is based on the fact that the natural map $\mathcal{M}(H_{g,1}^+) \cap \mathcal{M}(H_{g,1}^-) \rightarrow Aut(H_1(H_{g,1}^+))$ is onto. Therefore, the proof of [13] Lemma 4 works for this lemma. \square

We construct a subgroup of $\mathcal{M}(H_{g,1}^+) \cap \mathcal{M}(H_{g,1}^-)$ satisfying the above condition. Let $G \subset \mathcal{M}(H_{g,1}^+) \cap \mathcal{M}(H_{g,1}^-)$ be the subgroup generated by

$$\{h_i | i \in \{1, 2, \dots, g\}\} \cup \{s_{ij} | i \neq j\}$$

FIG 2. $c_{h,i}$ FIG 3. $c_{a,i}$, $c_{b,j}$ and $c_{i,j}$ for $j > i$ FIG 4. $c_{a,i}$, $c_{b,j}$ and $c_{i,j}$ for $i > j$

where we denote by h_i and s_{ij} the half twist along $c_{h,i}$ as in Figure 2 and the element $t_{c_{i,j}} t_{c_{a,i}}^{-1} t_{c_{b,j}}^{-1}$ as in Figure 3 and Figure 4. Since this subgroup G satisfies the condition in the above lemma, we have the following.

Lemma 2.5. *The equivalence relation \sim in $\mathcal{I}(\Sigma_{g,1})$ is generated by $\xi \sim \eta_G \xi \eta_G^{-1}$ for $\eta_G \in \{h_i, s_{i,j}\}$, $\xi \sim \xi \eta^+$ for $\eta^+ \in \{t_{\xi(c_a)\xi(c'_a)} | \xi \in \mathcal{M}(H_{g,1}^+)\} \cup \{t_{\xi(c_b)\xi(c'_b)} | \xi \in \mathcal{M}(H_{g,1}^+)\}$ and $\xi \sim \eta^- \xi$ for $\eta^- \in \{t_{\xi(c_a)\xi(c'_a)} | \xi \in \mathcal{M}(H_{g,1}^-)\} \cup \{t_{\xi(c_b)\xi(c'_b)} | \xi \in \mathcal{M}(H_{g,1}^-)\}$*

3. PROOF OF MAIN THEOREM

3.1. Completed Kauffman bracket skein algebras and Torelli groups. Let Σ be a compact connected oriented surface. We denote by $\mathcal{T}(\Sigma)$ the set of un-oriented framed tangles in $\Sigma \times I$. Let $\mathcal{S}(\Sigma)$ be the quotient of $\mathbb{Q}[A, A^{-1}]\mathcal{T}(\Sigma)$ by the skein relation and the trivial knot relation as in Figure 5. We consider the product of $\mathcal{S}(\Sigma)$ as in Figure 6 and the Lie bracket $[x, y] \stackrel{\text{def.}}{=} \frac{1}{-A+A^{-1}}(xy - yx)$ for $x, y \in \mathcal{S}(\Sigma)$. The completed Kauffman bracket skein algebra is defined by

$$\widehat{\mathcal{S}(\Sigma)} \stackrel{\text{def.}}{=} \varprojlim_{i \rightarrow \infty} \mathcal{S}(\Sigma) / (\ker \varepsilon)^i$$

where the augmentation map $\varepsilon : \mathcal{S}(\Sigma) \rightarrow \mathbb{Q}$ is defined by $A + 1 \mapsto 0$ and $[L] - (-2)^{|L|} \mapsto 0$ for $L \in \mathcal{T}(\Sigma)$. In [15], we define the filtration $\{F^n \widehat{\mathcal{S}(\Sigma)}\}_{n \geq 0}$ satisfying

$$\begin{aligned} F^n \widehat{\mathcal{S}(\Sigma)} F^m \widehat{\mathcal{S}(\Sigma)} &\subset F^{n+m} \widehat{\mathcal{S}(\Sigma)}, \\ [F^n \widehat{\mathcal{S}(\Sigma)}, F^m \widehat{\mathcal{S}(\Sigma)}] &\subset F^{n+m-2} \widehat{\mathcal{S}(\Sigma)}, \\ F^{2n} \widehat{\mathcal{S}(\Sigma)} &= (\ker \varepsilon)^n. \end{aligned}$$

By the second equation, we can consider the Baker Campbell Hausdorff series

$$\text{bch}(x, y) \stackrel{\text{def.}}{=} (-A + A^{-1}) \log(\exp(\frac{x}{-A + A^{-1}}) \exp(\frac{y}{-A + A^{-1}}))$$

on $F^3 \widehat{\mathcal{S}(\Sigma)}$. As elements of the associated Lie algebra $(\widehat{\mathcal{S}(\Sigma)}, [\ , \])$, bch has a usual expression. For example,

$$\text{bch}(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots.$$

Furthermore, we have the following.

Proposition 3.1 ([15] Corollary 5.7.). *For any embedding $i : \Sigma \times I \rightarrow S^3$ inducing $i_* : \widehat{\mathcal{S}(\Sigma)} \rightarrow \mathbb{Q}[[A + 1]]$, we have $i_*(F^n \widehat{\mathcal{S}(\Sigma)}) \subset ((A + 1)^{\lfloor \frac{n+1}{2} \rfloor})$, where $\lfloor x \rfloor$ is the greatest integer not greater than x for $x \in \mathbb{Q}$.*

In our previous papers [14] [15] [16], we study a relationship between the Kauffman bracket skein algebra and the mapping class group on a surface Σ . Let $\widehat{\mathcal{S}(\Sigma)}$ be the completed Kauffman bracket skein algebra on Σ and $\widehat{\mathcal{S}(\Sigma, J)}$ the completed Kauffman bracket skein module with base point set $J \times \{\frac{1}{2}\}$ for a finite subset $J \subset \partial\Sigma$. In [14], we prove the formula of the Dehn twist t_c of a simple closed curve c

$$t_c(\cdot) = \exp(\sigma(L(c))) (\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!} (\sigma(L(c))^i (\cdot) \in \text{Aut}(\widehat{\mathcal{S}(\Sigma, J)}))$$

where

$$L(c) \stackrel{\text{def.}}{=} \frac{-A + A^{-1}}{4 \log(-A)} (\text{arccosh}(\frac{-c}{2}))^2 - (-A + A^{-1}) \log(-A).$$

the skein relation

$$\bigcirc \text{ with cross} = A \bigcirc \text{ with arcs} + A^{-1} \bigcirc \text{ with arcs}$$

the trivial knot relation

$$\bigcirc \text{ with inner circle} = (-A^2 - A^{-2}) \bigcirc \text{ with inner dashed circle}$$

FIG 5. Definition of Kauffman bracket skein module

$$xy \stackrel{\text{def.}}{=} \begin{array}{|c|} \hline x \\ \hline y \\ \hline \end{array} \quad \begin{array}{c} 1 \\ 0 \end{array} \quad \Sigma$$

for $x, y \in \mathcal{S}(\Sigma)$

FIG 6. Definition of the product

We obtain the formula by analogy of the formula of the completed Goldman Lie algebra [6] [7] [9]. We define the filtration $\{F^n \widehat{\mathcal{S}(\Sigma)}\}_{n \geq 0}$ in [15]. We consider $F^3 \widehat{\mathcal{S}(\Sigma)}$ as a group using the Baker Campbell Hausdorff series bch where $g > 1$. We remark that $\mathcal{I}(\Sigma_{g,1})$ is generated by $\{t_{c_1 c_2} \stackrel{\text{def.}}{=} t_{c_1} t_{c_2}^{-1} | (c_1, c_2) : \text{BP}\}$ where a BP (bounding pair) is a pair of two simple closed curves bounding a submanifold of $\Sigma_{g,1}$. By analogy of [7] 6.3, we have the following.

Theorem 3.2 ([16] Theorem 3.13. Corollary 3.14.). *The group homomorphism $\zeta : \mathcal{I}(\Sigma_{g,1}) \rightarrow (F^3 \widehat{\mathcal{S}(\Sigma_{g,1})}, \text{bch})$ defined by $\zeta(t_{c_1 c_2}) = L(c_1) - L(c_2)$ for a BP (c_1, c_2) is injective whre $g > 1$. Furthermore, we have*

$$\xi(\cdot) = \exp(\sigma(\zeta(\xi))(\cdot)) \in \text{Aut}(\widehat{\mathcal{S}(\Sigma_{g,1}, J)})$$

for any $\xi \in \mathcal{I}(\Sigma_{g,1})$ and any finite subset $J \subset \partial \Sigma_{g,1}$.

We remark that $\zeta(t_c) = L(c)$ for a separating simple closed curve c .

Let e be an embedding $\Sigma_{g,1} \times [0, 1]$ satisfying the following conditions

$$\begin{aligned} e|_{\Sigma_{g,1} \times \{\frac{1}{2}\}} : \Sigma \times \{\frac{1}{2}\} &\rightarrow \Sigma, (x, \frac{1}{2}) \mapsto x, \\ e(\Sigma \times [0, \frac{1}{2}]) &\subset H_g^+, \\ e(\Sigma \times [\frac{1}{2}, 1]) &\subset H_g^-. \end{aligned}$$

We call this embedding a standard embedding. We denote by e_* the $\mathbb{Q}[[A+1]]$ -module homomorphism $\widehat{\mathcal{S}(\Sigma_{g,1})} \rightarrow \mathbb{Q}[[A+1]]$ induced by e . The following is our main theorem.

Theorem 3.3. *The map $Z : \mathcal{I}(\Sigma_{g,1}) \rightarrow \mathbb{Q}[[A+1]]$ defined by*

$$Z(\xi) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{(-A + A^{-1})^i i!} e_*((\zeta(\xi))^i)$$

induces

$$z : \mathcal{H}(3) \rightarrow \mathbb{Q}[[A+1]], M(\xi) \rightarrow Z(\xi).$$

3.2. Main theorem and its proof. The aim of this subsection is to prove Theorem 3.3.

By Proposition 3.1, the map $Z : \mathcal{I}(\Sigma_{g,1}) \rightarrow \mathbb{Q}[[A+1]]$ is well-defined.

For $\epsilon \in \{+, -\}$, let $\mathcal{S}(H_g^\epsilon)$ be the quotient of $\mathbb{Q}[A, A^{-1}]\mathcal{T}(H_g^\epsilon)$ by the skein relation and the trivial knot relation, where $\mathcal{T}(H_g^\epsilon)$ is the set of unoriented framed link in H_g^ϵ . We can consider its completion $\widehat{\mathcal{S}(H_g^\epsilon)}$, for details see [14] Theorem 5.1. We denote the embedding $\iota^+ : \Sigma_{g,1} \times [0, \frac{1}{2}] \rightarrow H_g^+$ and the embedding $\iota^- : \Sigma_{g,1} \times [\frac{1}{2}, 1] \rightarrow H_g^-$. The embeddings

$$\begin{aligned} \iota^+ : \Sigma_{g,1} \times I &\rightarrow H_g^+, (x, t) \mapsto \iota^+(x, t/2), \\ \iota^- : \Sigma_{g,1} \times I &\rightarrow H_g^-, (x, t) \mapsto \iota^+(x, (t+1)/2) \end{aligned}$$

induces

$$\iota^+ : \widehat{\mathcal{S}(\Sigma_{g,1})} \rightarrow \widehat{\mathcal{S}(H_g^+)}, \quad \iota^- : \widehat{\mathcal{S}(\Sigma_{g,1})} \rightarrow \widehat{\mathcal{S}(H_g^-)}.$$

By definition, we have the followings.

Proposition 3.4. (1) *The kernel of ι^+ is a right ideal of $\widehat{\mathcal{S}(\Sigma_{g,1})}$.*

(2) *The kernel of ι^- is a left ideal of $\widehat{\mathcal{S}(\Sigma_{g,1})}$.*

(3) *We have $e_*(\ker \iota^\epsilon) = \{0\}$ for $\epsilon \in \{+, -\}$.*

Proposition 3.5. *We have $Z(\xi_1 \xi_2) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{i+j} i! j!} e_*(((\zeta(\xi_1))^i (\zeta(\xi_2))^j)$.*

Lemma 3.6. (1) *Let c_a, c'_a, c_b and c'_b be simple closed curves as in Figure 1. For $\epsilon \in \{+, -\}$, we have*

$$\iota^\epsilon(\xi(L(c_a) - L(c'_a))) = 0, \iota^\epsilon(\xi(L(c_b) - L(c'_b))) = 0$$

for $\xi \in \mathcal{M}(H_{g,1}^\epsilon)$.

- (2) Let $c_{a,i}$, $c_{b,j}$ and $c_{i,j}$ be simple closed curves as in Figure 3 or Figure 4. For $\epsilon \in \{+, -\}$, we have

$$\iota^\epsilon(L(c_{i,j}) - L(c_{a,i}) - L(c_{b,j})) = 0$$

for $i \neq j$.

By Lemma 2.5, in order to prove Theorem 3.3, it is enough to check the following lemmas.

Lemma 3.7. *For any i , we have $e_* \circ h_i = e_*$. Furthermore, we have $Z(h_i \xi h_i^{-1}) = Z(\xi)$.*

Proof. The embeddings $e \circ h_i$ and e are isotopic. This proves the first claim. Using this, we have $e_*(\zeta(h_i \xi h_i^{-1})) = e_* \circ h_i(\zeta(\xi))$. This proves the second claim. This proves the lemma. \square

Lemma 3.8. *For any $i \neq j$, we have $e_* \circ s_{ij} = e_*$. Furthermore, we have $Z(s_{ij} \xi s_{ij}^{-1}) = Z(\xi)$.*

Proof. We fix an element x of $\widehat{\mathcal{S}(\Sigma_{g,1})}$. We have $s_{ij}(x) = \exp(\sigma(L(c_{i,j}) - L(c_{a,i}) - L(c_{b,j}))) (x)$. Using Lemma 3.6(2) and Proposition 3.4 (1)(2)(3), we have $e_*(\exp(\sigma(L(c_{i,j}) - L(c_{a,i}) - L(c_{b,j}))) (x)) = e_*(x)$. This proves the first claim. Using this, we have $e_*(\zeta(s_{ij} \xi s_{ij}^{-1})) = e_* \circ s_{ij}(\zeta(\xi))$. This proves the second claim. This proves the lemma. \square

Lemma 3.9. (1) *We have $Z(\xi \eta^+) = Z(\xi)$ for any $\xi \in \mathcal{I}(\Sigma_{g,1})$ and any $\eta^+ \in \{t_{\xi(c_a)\xi(c'_a)} | \xi \in \mathcal{M}(H_{g,1}^+)\} \cup \{t_{\xi(c_b)\xi(c'_b)} | \xi \in \mathcal{M}(H_{g,1}^+)\}$.*
 (2) *We have $Z(\eta^- \xi) = Z(\xi)$ for any $\xi \in \mathcal{I}(\Sigma_{g,1})$ and any $\eta^- \in \{t_{\xi(c_a)\xi(c'_a)} | \xi \in \mathcal{M}(H_{g,1}^-)\} \cup \{t_{\xi(c_b)\xi(c'_b)} | \xi \in \mathcal{M}(H_{g,1}^-)\}$.*

Proof. We prove only (1) (because the proof of (2) is almost the same.) By Proposition 3.5, we have $Z(\xi \eta^+) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{i+j} i! j!} e_*((\zeta(\xi))^i (\zeta(\eta^+))^j)$. Using Lemma 3.6 and Proposition 3.4 (1)(3), we obtain

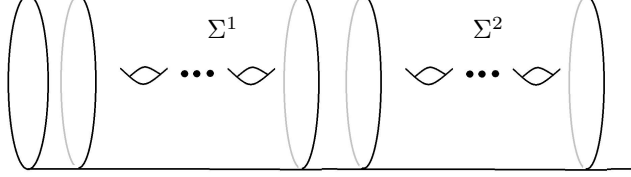
$$Z(\xi \eta^+) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{i+j} i! j!} e_*((\zeta(\xi))^i (\zeta(\eta^+))^j) = \sum_{i \geq 0} \frac{1}{(-A + A^{-1})^i i!} e_*((\zeta(\xi))^i) = Z(\xi).$$

This proves the lemma. \square

Proof of Theorem 3.3. By Fact 2.2 and Lemma 2.5, it is enough to check the following

$$\begin{aligned} Z(h_i \xi h_i^{-1}) &= Z(\xi), \\ Z(s_{ij} \xi s_{ij}^{-1}) &= Z(\xi), \\ Z(\xi \eta^+) &= Z(\xi), \\ Z(\eta^- \xi) &= Z(\xi), \end{aligned}$$

for any $\xi \in \mathcal{I}(\Sigma_{g,1})$, any $\eta^+ \in \{t_{\xi(c_a)\xi(c'_a)} | \xi \in \mathcal{M}(H_{g,1}^+)\} \cup \{t_{\xi(c_b)\xi(c'_b)} | \xi \in \mathcal{M}(H_{g,1}^+)\}$, any $\eta^- \in \{t_{\xi(c_a)\xi(c'_a)} | \xi \in \mathcal{M}(H_{g,1}^-)\} \cup \{t_{\xi(c_b)\xi(c'_b)} | \xi \in \mathcal{M}(H_{g,1}^-)\}$ and any $i \neq j$. The above lemmas prove these equations. This proves the Theorem. \square

FIG 7. Σ^1 and Σ^2

This invariant satisfies the following conditions.

Proposition 3.10. *For $M_1, M_2 \in \mathcal{H}(3)$, we have*

$$z(M_1 \sharp M_2) = z(M_1)z(M_2)$$

where $M_1 \sharp M_2$ is the connected sum of M_1 and M_2 .

Proof. Let $\iota_1 : \Sigma^1 \rightarrow \Sigma_{g,1}$ and $\iota_2 : \Sigma^2 \rightarrow \Sigma_{g,1}$ be the embedding maps as in Figure 7. The embedding maps induces

$$\begin{aligned} \iota_1 : \mathcal{M}(\Sigma^1) &\rightarrow \mathcal{M}(\Sigma_{g,1}), & \iota_2 : \mathcal{M}(\Sigma^2) &\rightarrow \mathcal{M}(\Sigma_{g,1}), \\ \iota_1 : \mathcal{S}(\Sigma^1) &\rightarrow \mathcal{S}(\Sigma_{g,1}), & \iota_2 : \mathcal{S}(\Sigma^2) &\rightarrow \mathcal{S}(\Sigma_{g,2}). \end{aligned}$$

We remark $e_*(\iota_1(x_1)\iota_2(x_2)) = e_*(\iota_1(x_1))e_*(\iota_2(x_2))$ for $x_1 \in \mathcal{S}(\Sigma^1)$ and $x_2 \in \mathcal{S}(\Sigma^2)$. For $\xi_1 \in \iota_1(\mathcal{M}(\Sigma^1))$ and $\xi_2 \in \iota_2(\mathcal{M}(\Sigma^2))$ we have

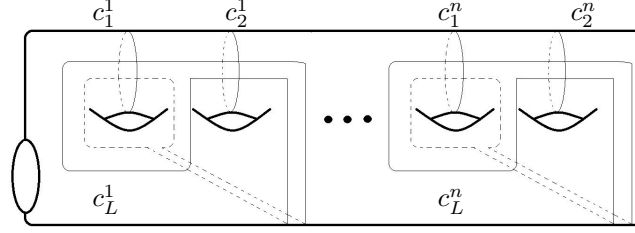
$$\begin{aligned} z(M(\xi_1) \sharp M(\xi_2)) &= z(M(\xi_1 \circ \xi_2)) = Z(\xi_1 \circ \xi_2) = \sum_{i=1}^{\infty} \frac{1}{(-A + A^{-1})^i i!} e_*((\zeta(\xi_1 \circ \xi_2))^i) \\ &= \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{i+j} i! j!} e_*((\zeta(\xi_1))^i (\zeta(\xi_2))^j) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{i+j} i! j!} e_*((\zeta(\xi_1))^i) e_*((\zeta(\xi_2))^j) \\ &= \left(\sum_{i=0}^{\infty} \frac{1}{(-A + A^{-1})^i i!} e_*((\zeta(\xi_1))^i) \right) \left(\sum_{j=0}^{\infty} \frac{1}{(-A + A^{-1})^j j!} e_*((\zeta(\xi_2))^j) \right) = z(M(\xi_1))z(M(\xi_2)). \end{aligned}$$

This proves the proposition. □

Proposition 3.11. *For $\xi_1 \in \zeta^{-1}(F^{n_1+2}\widehat{\mathcal{S}(\Sigma_{g,1})})$, \dots , $\xi_k \in \zeta^{-1}(F^{n_k+2}\widehat{\mathcal{S}(\Sigma_{g,1})})$, we have*

$$\sum_{\epsilon_i \in \{1,0\}} (-1)^{\sum \epsilon_i} z(M(\xi_1^{\epsilon_1} \dots \xi_k^{\epsilon_k})) \in (A+1)^{\lfloor (n_1 + \dots + n_k + 1)/2 \rfloor} \mathbb{Q}[[A+1]].$$

We remark that $\zeta^{-1}(F^3\widehat{\mathcal{S}(\Sigma_{g,1})})$ equals $\mathcal{I}(\Sigma_{g,1})$ and that $\zeta^{-1}(F^4\widehat{\mathcal{S}(\Sigma_{g,1})})$ equals the Johnson kernel.

FIG 8. $c_1^1, c_2^1, c_L^1, \dots, c_1^n, c_2^n, c_L^n$

Proof. We have

$$\begin{aligned} & \sum_{\epsilon_i \in \{1,0\}} (-1)^{\sum \epsilon_i} z(M(\xi_1^{\epsilon_1} \dots \xi_k^{\epsilon_k})) \\ &= e_*((1 - \exp(\frac{\zeta(\xi_1)}{-A + A^{-1}})) \dots (1 - \exp(\frac{\zeta(\xi_k)}{-A + A^{-1}}))). \end{aligned}$$

By Proposition 3.1, we have

$$\sum_{\epsilon_i \in \{1,0\}} (-1)^{\sum \epsilon_i} z(M(\xi_1^{\epsilon_1} \dots \xi_k^{\epsilon_k})) \in (A+1)^{\lfloor (n_1 + \dots + n_k + 1)/2 \rfloor} \mathbb{Q}[[A+1]].$$

This proves the proposition. \square

Corollary 3.12. *The invariant $z(M) \in \mathbb{Q}[[A+1]]/((A+1)^{n+1})$ is a finite type invariant for $M \in \mathcal{H}(3)$ order n .*

Proposition 3.13. *The invariant $z(M) \in \mathbb{Q}[[A+1]]/((A+1)^{n+1})$ is a nontrivial finite type invariant for $M \in \mathcal{H}(3)$ order n .*

Proof. It is enough to show that there exists $\xi_1, \dots, \xi_n \in \mathcal{K}(\Sigma_{g,1})$ such that

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} M(\prod_{i=1}^n \xi_i^{\epsilon_i}) \neq 0 \pmod{(A+1)^{n+1}}.$$

Let $c_1^1, c_2^1, c_L^1, \dots, c_1^n, c_2^n, c_L^n$ be simple closed curves as in Figure 8. We denote $c_i \stackrel{\text{def.}}{=} t_{c_1^i} \circ t_{c_2^i}(c_L^i)$ and $t_i \stackrel{\text{def.}}{=} t_{c_i}$ for $i = 1, \dots, n$. We remark that $M(t_{i_1} \circ \dots \circ t_{i_k}) = \sharp^k M(t_1)$ for $1 \leq i_1 < \dots < i_k \leq n$ and that $M(t_1)$ is the Poincaré homology 3-sphere. By the computation in section 4 and Proposition 3.10,

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} M(\prod_{i=1}^n t_i^{\epsilon_i}) = (1 - z(M(t_1)))^n = 6^n (A^4 - 1)^n \pmod{(A+1)^{n+1}}.$$

This proves the proposition. \square

By the computation in section 4, the coefficient of $A^4 - 1$ in the invariant z for the Poincaré homology 3-sphere is -6 . Since the casson invariant is the unique nontrivial finite type invariant of order 1 upto a scalar, we have the following.

Corollary 3.14. *For any $M \in \mathcal{H}(3)$, we have the coefficeint of $A^4 - 1$ in $z(M)$ is (-6) times the Casson invariant.*

4. EXAMPLE

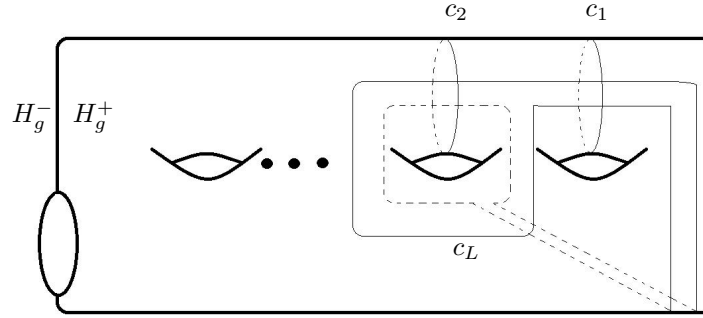
Let c_1, c_2 and c_L be simple closed curves in $\Sigma_{g,1}$ as in Figure 9. We consider integral homology 3-spheres $M(\epsilon_1, \epsilon_2, \epsilon_3) \stackrel{\text{def.}}{=} M((t_{t_1 \epsilon_1} \circ t_{t_2 \epsilon_2}(c_L))^{\epsilon_3})$ for $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$, where $t_1 \stackrel{\text{def.}}{=} t_{c_1}$ and $t_2 \stackrel{\text{def.}}{=} t_{c_2}$. For $\epsilon \in \{\pm 1\}$, the manifold $M(1, -1, \epsilon) \simeq M(-1, 1, \epsilon)$ is the integral homology 3-sphere obtained from S^3 performing the ϵ -surgery on the figure eight knot 4_1 , which is $e(t_1^{-1} \circ t_2(c_L))$. For $\epsilon \in \{\pm 1\}$, the manifold $M(1, 1, \epsilon)$ is the integral homology 3-sphere obtained from S^3 performing the ϵ -surgery on the trefoil knot 3_1 , which is $e(t_1 \circ t_2(c_L))$. For $\epsilon \in \{\pm 1\}$, the manifold $M(-1, -1, \epsilon)$ is the integral homology 3-sphere obtained from S^3 performing the ϵ -surgery on the mirror -3_1 of the trefoil knot, which is $e(t_1^{-1} \circ t_2^{-1}(c_L))$. In particular, $M(1, 1, 1)$ is the Poincaré homology sphere. We remark that $M(1, -1, 1)$ and $M(-1, -1, -1)$ are the same 3-manifold. By straight forward computations using Habiro's formula [4] for colored Jones polynomials of the trefoil knot and the figure eight knot, we have the following. We remark that we compute $z(M(1, -1, 1)) = z(M(1, 1, -1))$ (rep. $z(M(1, -1, -1)) = z(M(-1, -1, 1))$) by two ways $Z(t_{t_1 \circ t_2^{-1}(c_L)}) = Z((t_{t_1 \circ t_2}(c_L))^{-1})$ (resp. $Z((t_{t_1 \circ t_2^{-1}(c_L)})^{-1}) = Z(t_{t_1^{-1} \circ t_2^{-1}(c_L)})$).

Proposition 4.1. *We have*

$$\begin{aligned}
z(M(1, 1, 1)) &= [1, -6, 45, -464, 6224, -102816, 2015237, \\
&\quad -45679349, 1175123730, -33819053477, \\
&\quad 1076447743008, -37544249290614, \\
&\quad 1423851232935885, -58335380481272491], \\
z(M(1, -1, 1)) &= z(M(1, 1, -1)) = [1, 6, 63, 932, 17779, 415086, 11461591, 365340318, \\
&\quad 13201925372, 533298919166, 23814078531737, \\
&\quad 1164804017792623, 61932740213389942, \\
&\quad 3556638330023177088], \\
z(M(-1, -1, 1)) &= [1, 6, 39, 380, 4961, 80530, 1558976, 35012383, \\
&\quad 894298109, 25591093351, 810785122236, \\
&\quad 28169720107881, 1064856557864671, \\
&\quad 43506118030443092], \\
z(M(1, -1, -1)) &= z(M(-1, -1, 1)) = [1, -6, 69, -1064, 20770, -492052, 13724452, \\
&\quad -440706098, 16015171303, -649815778392, 29121224693198, \\
&\quad -1428607184648931, 76147883907835312, \\
&\quad -4382222160786508572].
\end{aligned}$$

Here we denote

$$1 + a_1(A^4 - 1) + a_2(A^4 - 1)^2 + \cdots + a_{13}(A^4 - 1)^{13} + o(14) = [1, a_1, a_2, \cdots, a_{13}]$$

FIG 9. simple closed curves c_1 , c_2 and c_L

where $o(14) \in (A+1)^{14}\mathbb{Q}[[A+1]]$.

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